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# THE CONSTRUCTION OF COMPOSITE MODELS FOR THE MECHANICS OF A SUPPORT OF A SET OF RIGID BODIES*

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Previous results /1, 2/, based on /3, 4/, were used to devise universal and economic algorithms for constructing composite matrix models for the mechanics (kinematics, dynamics and control) of the support of a set of rigid bodies with a tree structure /5-10/. Two types of model were considered, which either take account of, or ignore, the effect of support motion on the relative motion of the supported bodies.

The models function well because, at the first stage of their formation, dynamic sets (equations of dynamics, written in a special form in quasivelocities) of individual sections of the system (i.e., rigid bodies of the same type) are constructed. By combining these equations in matrix form, a single vector equation of motion of the system is obtained, when the system is under the action of external forces, control forces, friction forces, and reactions of the hinges. Multiplication of the equation by the structural matrix determined by the graph and instantaneous configuration of the system of bodies, leads to an equation of the support motion that takes account of its effect on the relative motion of the supported bodies.

The matrix coefficients of these equations prove to be obtainable by a certain rule (given by the graph of the system) from the similar matrix coefficients of the equations of motion of the isolated bodies. This makes the models suitable for computer construction in symbolic form /11, 12/, and/or, independently, in numerical form according to a simple rule which is easily formalized and therefore readily adapted for computer use. These models embrace any class of kinematic pairs of supported systems, the presence on the bodies of dynamically unbalanced and asymmetric rotating flywheels, and the presence of an external delay medium (in the context of potential flow).

**1.** All our working is based on the following concepts /1-4/.

1°. The numerical set of vector space of screws $H$ and the group of its motions

$$L(H.6) = \{L_t^s : L_t^s = T_t^s[c_t^s];\ s, t \in N\}$$
$$L_t^s = L_{s+1}^s L_{s+2}^{s+1} \times \ldots \times L_t^{t-1}, \quad (L_t^s)^{-1} = L_s^t \tag{1.1}$$

Here and henceforth, the notation is the same as that introduced in /2, 3/.

The group $L(H.6)$ is convenient for computerizing the matrix formalization of repeated operations of the transformation of screw Plücker coordinates when changing to a new system of coordinates: if $X_s^s$ and $X_t^t$ are the same screw $X \in H$ in $E_s$ and $E_t$, where $L_t^s$: $E_s \to E_t$, then $X_s^s = L_t^s X_t^t$, $X_t^t = L_s^t X_s^s$.

2°. The kinematic equation in the group $L(H.6)$

$$L_t^{s\cdot} = L_t^s \Phi_t^{st}, \quad \Phi_t^{st} = \begin{Vmatrix} \langle \omega_t^s \rangle^t & 0 \\ \langle \vartheta_t^s \rangle^t & \langle \omega_t^s \rangle^t \end{Vmatrix} \tag{1.2}$$

Eq.(1.2) is convenient for computerizing the matrix formalization of differentiation of motions $L_t^s \in L(H.6)$.

3°. The recurrent kinematic equations

$$V_t^{st} = L_t^{s+1,\,T} V_{s+1}^{s.s+1} + V_t^{s+1.t}$$

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$$\Phi_t^{st} = L_{s+1}^t \Phi_{s+1}^{s.s+1} L_t^{s+1} + \Phi_t^{s+1.t} \tag{1.3}$$
$$L_{s+1}^t,\ L_t^{s+1} \in L(H.6),\quad V_t^{st} = \| V_t^{st}, \omega_t^{st} \|^T$$

4°. The tree-type graph of the system, whose orientation is determined by the vectors of the configuration $E_{lk}$ in $E_{\mu.k-1}$, $\mu \leqslant l$ ($\mu$, $l$ are the column numbers, and $k$ is the number of levels of the graphs, for a support-body $l = 1$, $k = 1$)

$$\mathbf{r}^{lk} = \| s^{lk}, \theta^{lk} \|^T \tag{1.4}$$

where $s^{lk} = \| s_1^{lk}, s_2^{lk}, s_3^{lk} \|^T$ is the vector of parallel translation $E_{lk}$ in $E_{\mu.k-1}$ and in the basis $[e^{\mu.k-1}]$; $\theta^{lk} = \| \theta_4^{lk}, \theta_5^{lk}, \theta_6^{lk} \|^T$ is the vector of angles of orientation of $[e^{lk}]$ in $[e^{\mu.k-1}]$.

In the case of constructive (constant) translations and rotations, we shall use, instead of the symbols $s$ and $\theta$ the symbols $p$ and $\varphi$ respectively. The variable coordinates of the vector (1.4) form the vector of generalized coordinates $\mathbf{q}^{lk}$ of the kinematic pair ($\mu.\ k-1$; $lk$), so that

$$\mathbf{r}^{lk\cdot} = \| f^{lk} \| \mathbf{q}^{lk\cdot} \tag{1.5}$$

where $\| f^{lk} \| = \| \ldots | f_\beta^{lk} | \ldots \|$ is a $(6 \times \dim q^{lk})$ matrix, whose columns are the six-dimensional unit vectors $f_\beta^{lk} \in R_6$ of the axes of mobility with one in the $\beta$-position, $\beta = 1, 2, \ldots, 6$.

5°. The structural matrix of the system is

$$S_M = \| f \|^T M^T L \tag{1.6}$$
$$\| f \| = \operatorname{diag}(\| f^{lk} \|),\quad M = \operatorname{diag}(M_{lk}^{\mu.k-1})$$
$$M_{lk}^{\mu.k-1} = \begin{Vmatrix} c_{lk}^{\mu.k-1,\,T} & 0 \\ 0 & \varepsilon_{lk}^{\mu.k-1} \end{Vmatrix} \tag{1.7}$$
$$c_{lk}^{\mu.k-1} = c_1(\theta_4^{lk})\, c_2(\theta_5^{lk})\, c_3(\theta_6^{lk}) \tag{1.8}$$
$$\varepsilon_{lk}^{\mu.k-1} = \| c_3^T(\theta_6^{lk})\, c_2^T(\theta_5^{lk})\, e_1^{lk} \,|\, c_3^T(\theta_6^{lk})\, e_2^{lk} \,|\, e_3^{lk} \| \tag{1.9}$$
$$c_i(\theta_\alpha^{lk}) = E + \langle e_i^{lk} \rangle \sin\theta_\alpha^{lk} + \langle e_i^{lk} \rangle^2 (1 - \cos\theta_\alpha^{lk}) \tag{1.10}$$

Here, $c_{lk}^{\mu.k-1}$ is the matrix of rotation $[e^{\mu.k-1}] \to [e^{lk}]$; $\varepsilon_{lk}^{\mu.k-1}$ is the matrix of the Euler kinematic equations, $\omega_{lk}^{\mu.k-1;\,lk} = \varepsilon_{lk}^{\mu.k-1} \theta^{lk\cdot}$; $c_i(\theta_\alpha^{lk})$ is a $(3 \times 3)$ matrix of elementary rotation with unit vector $e_i^{lk}$ by an angle $\theta_\alpha$; $i = 1, 2, 3$; $\alpha = 4, 5, 6$; $L$ is a $(6m \times 6m)$ upper-triangle block matrix with $(6 \times 6)$ blocks of the type

$$L_{lk}^{st} \in L(H.6), \ \text{if}\ (lk) \in (st)_+ \tag{1.11}$$
$$0, \ \text{if}\ (lk) \notin (st)_+$$

at the intersection of the $(st)$-matrix $(6 \times 6m)$ rows and the $(lk)$-matrix $(6m \times 6)$ columns; $m$ is the number of rigid bodies in the system: $(\cdot)_+$ is the set of attainability of the graph element $(\cdot)$.

The blocks of matrix $S_M$ are either zero numerical $(1 \times 6)$ rows, or columns of the same dimensionality of the type

$$s_{lk}^{st-\alpha} = f_\alpha^{st,\,T} M_{st}^{\mu.t-1,\,T} L_{lk}^{st} \tag{1.12}$$

Any screw $X_{lk}^{lk}$ is successively transformed by the action of this row from $E_{lk}$ into $E_{st}$, from $E_{st}$ into the system of generalized coordinates $q^{st}$, and then, by multiplication by the unit vector $f_\alpha^{st,T} \in R_6$, is projected onto the $\alpha$-direction of this coordinate system, $\alpha = 1, 2, \ldots, 6$.

2. The dynamic aggregate of a single rigid body is an element of the system which moves in an inertial fluid and supports dynamically unbalanced ($\langle r_c^{lk} \rangle^{lk} \neq 0$ in the von **Mises matrix** $\Theta_{lk}^{lk}$) **and asymmetric** ($\theta_{11}^{lk} \neq \theta_{22}^{lk}$ **in the matrix** $\Theta_{lk}^{lk}$) **flywheels, and can be written as** /2/

$$A_{lk}^{lk} V_{lk}^{10.lk*} + B_{lk}^{lk} V_{lk}^{10.lk} = F_{lk}^{lk} \tag{2.1}$$
$$A_{lk}^{lk} = \Theta_{lk}^{lk} + \Sigma_{s=1}^{m_{lk}} L_s^{lk} \Theta_s^s L_s^{lk,\,T} + \Lambda_{lk}^{lk}$$
$$B_{lk}^{lk} = \Phi_{lk}^{10.lk} A_{lk}^{lk} + A_{lk}^{lk*}$$
$$A_{lk}^{lk*} = \Sigma_{s=1}^{m_{lk}} L_s^{lk} ([\langle e_3^s \rangle]\, \Theta_s^{lk} - \Theta_s^{lk} [\langle e_3^s \rangle])\, L_s^{lk,\,T} \psi_s^{lk\cdot}$$
$$F_{lk}^{lk} = T_{lk}^{lk} + R_{lk}^{lk} + U_{lk}^{lk} + N_{lk}^{lk}$$

$$
\begin{aligned}
T_{lk}^{lk} &= P_{lk}^{lk} + \Sigma_{s=1}^{m_{kl}} L_s^{lk} P_s^{\,s} - \Sigma_{s=1}^{m_{lk}} (I_s^{lk} \psi_s^{lk\cdot\cdot} + J_s^{lk} \psi_s^{l\cdot\cdot}) \\
R_{lk}^{lk} &= R_{lk}^{lk}(\mu.k-1; lk) - L_{\nu.k+1}^{lk} R_{\nu.k+1}^{\nu.k+1}(lk; \nu.k+1) \\
U_{lk}^{lk} &= U_{lk}^{lk}(\mu.k-1; lk) - L_{\nu.k+1}^{lk} U_{\nu.k+1}^{\nu.k+1}(lk; \nu.k+1) \\
N_{lk}^{lk} &= N_{lk}^{lk}(\mu.k-1; lk) - L_{\nu.k+1}^{lk} N_{\nu.k+1}^{\nu.k+1}(lk; \nu.k+1)
\end{aligned}
$$

In Eqs. (2.1), $L_s^{lk} \in L\,(H.\ 6)$ is the matrix of passage from $E_{lk}$ to the coordinate system $E_s$, which is connected in a fixed way with the s-flywheel, mounted on the $(lk)$-section in the **power** module that controls the variation of the generalized coordinate $q_\beta^{\,lk}$ of the $(\nu, k+1)$-section $(\nu \geqslant l)$, $\Theta_s^{\,lk}$, $e_3^{\,s}$, $\psi_s^{\,lk}$ is the von Mises matrix, the unit vector of the axis of rotation and the angular velocity of rotation of the flywheel; $m_{lk}$ is the number of flywheels mounted on the $(lk)$-body for controlling the variation of the coordinate $q_\beta^{\,\nu.k+1}$; $R_{lk}^{\,lk}(\mu.\ k-1;\ lk)$; $U_{lk}^{\,lk}(\mu.\ k-1;\ lk)$, $N_{lk}^{\,lk}(\mu.\ k-1;\ lk)$ are the dynamic reaction screws that control the stresses and the friction stresses transmitted from the $(\mu.\ k-1)$-section to the $(lk)$-section in $E_{lk}$.

By combining Eqs. (2.1) (for all the system indices $(lk)$ we obtain the matrix equation of the system motion under the action of the external forces, the reaction forces at the hinges, and the control and friction stresses

$$
\begin{aligned}
&A\mathbf{V}^* + B\mathbf{V} = \mathbf{F} \\
&A = \mathrm{diag}\,(A_{lk}^{\;lk}), \quad B = \mathrm{diag}\,(B_{lk}^{\;lk}) \\
&\mathbf{F} = \|\ldots, F_{lk}^{lk}, \ldots, \|^T = \mathbf{T} + \mathbf{R} + \mathbf{U} + \mathbf{N} \\
&\mathbf{V} = \|\ldots, V_{lk}^{10.lk}, \ldots\|^T \\
&\mathbf{T} = \|\ldots, T_{lk}^{lk}, \ldots, \|^T, \quad \mathbf{R} = \|\ldots, R_{lk}^{lk}, \ldots\|^T \\
&\mathbf{U} = \|\ldots, U_{lk}^{lk}, \ldots\|^T, \quad \mathbf{N} = \|\ldots, N_{lk}^{lk}, \ldots\|^T
\end{aligned}
\tag{2.2}
$$

The equation of motion of the support, in quasivelocities, allowing for the influence of its motion on the relative motion of the supported bodies, is obtained by multiplying Eqs. (2.2) on the left by the structural matrix $S_M$ of (1.6):

$$
\begin{aligned}
&S_M A\mathbf{V}^* + S_M B\mathbf{V} = S_M\mathbf{T} + \mathbf{u} + \mathbf{n} \\
&S_M\mathbf{R} = 0, \quad S_M\mathbf{U} = \mathbf{u}, \quad S_M\mathbf{N} = \mathbf{n} \\
&\mathbf{u} = \|\ldots, u_\alpha^{\;st}, \ldots\|^T, \quad \mathbf{n} = \|\ldots, n_\alpha^{\;st}, \ldots\|^T
\end{aligned}
\tag{2.3}
$$

( $u_\alpha^{\;st}$, $n_\alpha^{\;st}$ are the control and friction stresses transmitted to the $(st)$-body with respect to the $\alpha$-coordinate).

Using Eqs. (1.4), we obtain for the kinematic aggregate of the system of bodies

$$
\mathbf{V} = S_M^T \mathbf{q}^{\cdot}, \quad \mathbf{q}^{\cdot} = \|\ldots, \mathbf{q}^{lk\cdot}, \ldots\|^T \tag{2.4}
$$

the differentiation of which gives

$$
\begin{aligned}
&\mathbf{V}^* = S_M^T \mathbf{q}^{\cdot\cdot} + S_M^{T\cdot} \mathbf{q}^{\cdot} \\
&S_M{}^{T\cdot} = (L^{T\cdot} M + L^T M^{\cdot})\, \| f \|
\end{aligned}
\tag{2.5}
$$

where the asterisk denotes the derivative in the connected coordinate systems.

The matrices $L^{\cdot}$ and $M^{\cdot}$ are obtained from matrices $L$ and $M$ by replacing the blocks $L_{lk}^{\;st}$ by $L_{lk}^{\;st\cdot} = L_{lk}^{\;st} \Phi_{lk}^{\;st.lk}$ of (1.2) and $M_{lk}^{\;\mu.k-1}$ of (1.7) by $M_{lk}^{\;\mu.lk\cdot}$, where

$$
\begin{aligned}
&M_{lk}^{\mu.k-1\cdot} = \left\| \begin{matrix} \langle \omega_{lk}^{\mu.k-1} \rangle^{lk,\,T} c_{lk}^{\mu.k-1,\,T} & 0 \\ 0 & \varepsilon_{lk}^{\mu.k-1\cdot} \end{matrix} \right\| \\
&\varepsilon_{lk}^{\mu.k-1\cdot} = \| (\langle e_3^{lk} \rangle^T c_3{}^T (\theta_6^{lk})\, c_2{}^T (\theta_5^{lk})\, \theta_6^{lk\cdot} + \\
&\qquad c_3{}^T (\theta_6^{lk})\, c_2{}^T (\theta_5^{lk}) \langle e_2^{lk} \rangle^T \theta_5^{lk\cdot})\, e_1^{lk} \,| \\
&| \langle e_3^{lk} \rangle^T c_3 (\theta_6^{lk})\, \theta_6^{lk\cdot} e_2^{lk} \,|\, 0 \,\|
\end{aligned}
\tag{2.6}
$$

When obtaining Eqs. (2.6) we use Poisson's equation $c_p^{\,p-1\cdot} = c_p^{\,p-1} \langle \omega_p^{\,p-1} \rangle^p = \langle \omega_p^{\,p-1} \rangle^p c_p^{\,p-1}$, under the condition that $c_p^{\,p-1}$ is a $(3 \times 3)$ matrix of an elementary rotation (1.10).

Substitution of (2.4) and (2.5) into (2.3) leads to the equation of the support motion when the influence of its motion on the relative motion of the supported bodies of generalized coordinates is taken into account

$$\mathbf{A}(\mathbf{q})\,\mathbf{q}^{\cdot\cdot} + \mathbf{B}(\mathbf{q},\mathbf{q}^{\cdot})\,\mathbf{q}^{\cdot} = S_M\mathbf{T} + \mathbf{u} + \mathbf{n} \tag{2.7}$$

$$\begin{aligned} &\mathbf{A}\,(q) = S_M^{-}\mathbf{A}S_M{}^T \\ &\mathbf{B}\,(q,q^{\cdot}) = S_M\mathbf{B}S_M{}^T + S_M\mathbf{A}S_M{}^{T^{\cdot}} \end{aligned} \tag{2.8}$$

By (2.7) and (2.8), the matrices $\mathbf{A}(\mathbf{q})$ and $\mathbf{B}(\mathbf{q},\mathbf{q}^{\cdot})$ of any system of bodies with a structural tree can be obtained from the similar matrices $A_{lk}{}^{lk}$, $B_{lk}{}^{lk}$ that form the system with the aid of the structural matrix $S_M$ and its derivative $S_M{}^{\cdot}$.

If the supported systems are all formed from kinematic pairs of the fifth class, then the matrix coefficients in (2.7) take the form (because $M_{lk}{}^{\mu.k-1}\,\|\,f^{lk}\,\| \equiv M_{lk}{}^{\mu.k-1}f_\beta{}^{lk} \equiv f_\beta{}^{lk}$, $M_{lk}{}^{\mu.k-1^{\cdot}}f_\beta{}^{lk} \equiv 0,\ lk \neq 11$)

$$\begin{aligned} &\mathbf{A}\,(\mathbf{q}) = SAS^TM_{11}, \quad S = \|\,f\,\|^TL \\ &\mathbf{B}\,(\mathbf{q},\mathbf{q}^{\cdot}) = SBS^TM_{11} + SAS_M{}^{T^{\cdot}} \\ &S_M{}^{T^{\cdot}} = S^{T^{\cdot}}M_{11} + S^TM_{11}{}^{\cdot} \\ &M_{11} = \|\,f\,\|^TM\,\|\,f\,\|, \quad M_{11}{}^{\cdot} = \|\,f\,\|^TM^{\cdot}\,\|\,f\,\| \end{aligned} \tag{2.9}$$

The block diagonal matrices $M_{11}$, $M_{11}{}^{\cdot}$ have the $(6\times 6)$ matrices $M_{11}{}^{10}$ and $M_{11}{}^{10}$ as the upper left-hand blocks. At the remaining places of the principal diagonals there are scalar ones and zeros.

Relations (2.8) and (2.9) give the matrices $\mathbf{A}\,(\mathbf{q})$ and $\mathbf{B}\,(\mathbf{q},\mathbf{q}^{\cdot})$ in Eqs.(2.7) as the products of block matrices. In many problems these matrices are better given by algorithms for obtaining their elements at the intersection of the $(st-\alpha)$-rows and the $(lk-\beta)$-columns

$$A_{lk-\beta}^{st-\alpha} = \sum_{p\in(st,\,lk)_+} s_p^{st-\alpha}A_p{}^ps_p^{lk-\beta,\,T} \tag{2.10}$$

$$\begin{aligned} &B_{lk-\beta}^{st-\alpha} = \sum_{p\in(st,\,lk)_+}(s_p^{st-\alpha}A_p{}^pc_p^{lk-\beta} + s_p^{st-\alpha}K_p^{lk.p}s_p^{lk-\beta,\,T}) \\ &s_p^{st-\alpha} = f_\alpha^{st,\,T}M_{st}^{\mu.t-1}L_p{}^{st},\ c_p^{lk-\beta} = L_p^{lk,\,T}M_{lk}^{\mu.k-1^{\cdot}}f_\beta^{lk} \\ &K_p^{lk.p} = B_p{}^p + A_p{}^p\Phi_p^{lk.p,\,T}, \quad (st,lk)_+ = (st)_+\cap(lk)_+ \\ &A_{lk-\beta}^{st-\alpha} = B_{lk-\beta}^{st-\alpha} = 0, \quad (st,lk)_+ = \varnothing \end{aligned} \tag{2.11}$$

For systems of supported bodies $(lk \neq 11)$ consisting of kinematic pairs of the fifth class, we obtain

$$\begin{aligned} &A_{lk-\beta}^{st-\alpha} = \sum_{p\in(st,\,lk)_+} s_p^{st-\alpha}A_p{}^ps_p{}^{lk-\beta,\,T} \\ &B_{lk-\beta}^{st-\alpha} = \sum_{p\in(st,\,lk)_+} s_p^{st-\alpha}K_p^{lk.p}s_p{}^{lk-\beta,\,T} \end{aligned} \tag{2.12}$$

3. From Eqs.(2.7), using Eqs.(2.10) and (2.11), we can find the matrix equations of the support motion when no account is taken of the influence of its motion on the motion of the supported bodies.

These equations are the first six scalar equations of (2.7), written in matrix form on the assumption that $\|\,f^{11}\,\| \equiv E$ is a $(6\times 6)$ identity matrix, $u^{11}=0,\ n^{11}=0$, after cancelling by the left-handed factor, common to all terms, $M_{11}{}^{10,T}$ $(\det M_{11}{}^{10} \neq 0)$,

$$\mathbf{A}(\mathbf{q},\gamma)\,\mathbf{q}^{\cdot\cdot} + \mathbf{B}(\mathbf{q},\gamma,\mathbf{q}^{\cdot},\gamma^{\cdot})\,\mathbf{q}^{\cdot} + \mathbf{I}(\mathbf{q},\gamma)\,\gamma^{\cdot\cdot} + \mathbf{J}(\mathbf{q},\gamma,\mathbf{q}^{\cdot},\gamma^{\cdot})\,\gamma^{\cdot} = L_+{}^{11}\mathbf{T} \tag{3.1}$$

$$\begin{aligned} &\mathbf{A}(\mathbf{q},\gamma) = A_+{}^{11}M_{11}{}^{10} \\ &B(\mathbf{q},\gamma,\mathbf{q}^{\cdot},\gamma^{\cdot}) = K_+{}^{11}M_{11}{}^{10} + A_+{}^{11}M_{11}^{10^{\cdot}} \\ &\mathbf{I}(\mathbf{q},\gamma) = \|\ldots|\,I_\beta^{lk}\,|\ldots\|, \quad \mathbf{J}(\mathbf{q},\gamma,\mathbf{q}^{\cdot},\gamma^{\cdot}) = \|\ldots|\,J_\beta^{lk}\,|\ldots\| \\ &I_\beta^{lk} = L_{lk}^{11}A_+^{lk}M_{lk}^{\mu.k-1}f_\beta^{lk} \\ &J_\beta^{lk} = L_{lk}^{11}(K_+^{lk}M_{lk}^{\mu.k-1} + A_+^{lk}M_{lk}^{\mu.k-1^{\cdot}})\,f_\beta^{lk} \\ &A_+^{lk} = \sum_{p\in(lk)_+} L_p^{lk}A_p{}^pL_p^{lk,\,T}, \quad K_p^{lk.p} = B_p{}^p + A_p{}^p\Phi_p^{lk.p,\,T} \\ &K_+^{lk} = \sum_{p\in(lk)_+} L_p^{lk}K_p^{lk.p}L_p^{lk,\,T} \\ &L_+{}^{11} = \|\,E\,|\,L_{12}{}^{11}\,|\,L_{13}{}^{11}\,|\ldots\| \\ &\mathbf{q}\equiv\mathbf{q}^{11}, \quad \gamma\equiv\mathbf{q}_+{}^{12} = \|\,\mathbf{q}^{12},\mathbf{q}^{13},\ldots\|^T \end{aligned}$$

Using the second of Eqs.(1.3), we can write the matrix $\mathbf{B}\,(\mathbf{q},\gamma,\mathbf{q}^{\cdot},\gamma^{\cdot})$ in (3.1) as

$$\mathbf{B}(\mathbf{q}, \boldsymbol{\gamma}, \mathbf{q}^{\cdot}, \boldsymbol{\gamma}^{\cdot}) = \Phi_{11}^{10\cdot 11} A_{+}{}^{11} M_{11}{}^{10} + A_{+}{}^{11} M_{11}^{10^{\cdot}} + \sum_{p \in (11)_+} L_p{}^{11} (\Phi_p^{11\cdot p} A_p{}^p + A_p{}^p \Phi_p^{11\cdot p,\,T} + A_p^{p*}) L_p^{11,\,T} M_{11}{}^{10} \tag{3.2}$$

For supported systems with kinematic pairs of the fifth class, the matrix coefficients are more simply written as

$$I_\beta^{lk} = L_{lk}^{11} A_+^{lk} f_\beta^{lk}, \quad J_\beta^{lk} = L_{lk}^{11} K_+^{lk} f_\beta^{lk} \tag{3.3}$$

Notice that (3.1) is an, in principle, new equation of motion of the support, which is simpler than Lagrange's equation of the second kind (the matrix $\mathbf{A}(\mathbf{q}, \boldsymbol{\gamma})$ is not the kinetic energy matrix of the "congealed" system, which has the form $\mathbf{A}(\mathbf{q}, \boldsymbol{\gamma}) = M_{11}{}^{10,T} A_+{}^{11} M_{11}{}^{10}$; and the vector $L_+{}^{11}\mathbf{T}$ is not the vector of generalized forces, referred to the coordinates $\mathbf{q}$, which has the form $M_{11}{}^{10,T} L_+{}^{11}\mathbf{T}$).

When obtaining Eqs.(3.1), we have renumbered the indices of the bodies of the system. We can renumber, using the same indices, the coordinate systems which participate in just one elementary motion (translation or rotation), by introducing for the appropriate indices fictitious inertialess bodies. In this case, the coefficients $I_\beta{}^{lk}$ and $J_\beta{}^{lk}$ for the supported systems with kinematic pairs of any class, will again be given by the simple relations (3.3). It is not **obvious** that the choice of these coordinate systems is desirable. On the one hand, the algorithms of (3.1) take the simpler form (3.3), so that the programming problem is simplified; but on the other hand, there are more matrix factors in the matrices $L_p{}^{p-1} \in L(H.6)$, **and** hence in the matrices $A_+{}^{lk}$ and $K_+{}^{lk}$. The question needs to be specially studied in each special case (depending on the graph structure, and the number of kinematic pairs of any class).

4. With the aid of Eqs.(3.1) we can solve direct and inverse problems of the support dynamics.

1°. Given the relative motion of the supported system $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t)$, $\boldsymbol{\gamma}^{\cdot} = \boldsymbol{\gamma}^{\cdot}(t)$, $\boldsymbol{\gamma}^{\cdot\cdot} = \boldsymbol{\gamma}^{\cdot\cdot}(t)$. The support motion $\mathbf{q}(t)$ is found by integrating the equation

$$\mathbf{A}(\mathbf{q}, \boldsymbol{\gamma}(t))\,\mathbf{q}^{\cdot\cdot} + \mathbf{B}(\mathbf{q}, \boldsymbol{\gamma}(t), \mathbf{q}^{\cdot}, \boldsymbol{\gamma}^{\cdot}(t))\,\mathbf{q}^{\cdot} = L_+{}^{11}\mathbf{T} - \mathbf{I}(\mathbf{q}, \boldsymbol{\gamma}(t))\,\boldsymbol{\gamma}^{\cdot\cdot}(t) - \mathbf{J}(\mathbf{q}, \boldsymbol{\gamma}(t), \mathbf{q}^{\cdot}, \boldsymbol{\gamma}^{\cdot}(t))\,\boldsymbol{\gamma}^{\cdot}(t) \tag{3.4}$$

2°. Suppose we are given the support motion $\mathbf{q}(t)$, $\mathbf{q}^{\cdot}(t)$ and $\mathbf{q}^{\cdot\cdot}(t)$. The class of relative motions of the system, such that this support motion is obtained (if it exists), is found by means of the equation

$$\mathbf{I}(\mathbf{q}(t), \boldsymbol{\gamma})\,\boldsymbol{\gamma}^{\cdot\cdot} + \mathbf{J}(\mathbf{q}(t), \boldsymbol{\gamma}, \mathbf{q}^{\cdot}(t), \boldsymbol{\gamma}^{\cdot})\,\boldsymbol{\gamma}^{\cdot} = L_+{}^{11}\mathbf{T} - \mathbf{A}(\mathbf{q}(t), \boldsymbol{\gamma})\,\mathbf{q}^{\cdot\cdot}(t) - \mathbf{B}(\mathbf{q}(t), \boldsymbol{\gamma}, \mathbf{q}^{\cdot}(t), \boldsymbol{\gamma}^{\cdot})\,\mathbf{q}^{\cdot}(t)$$

If $\xi = \dim \boldsymbol{\gamma} > 6$, we can divide the components of the vector $\boldsymbol{\gamma}$ into six dependent $\boldsymbol{\gamma}_+ \in R_6$, and $\xi - 6$ independent $\boldsymbol{\gamma}_- \in R_{\xi-6}$, components, i.e., $\boldsymbol{\gamma} = \|\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-\|^T$, and thus state the optimal control problem (similar to (13) in /2/). We can formulate the problem of finding the mass-inertial characteristics of the supported systems or their structures, required to solve the control problem.

5. The following facts ensure efficient composite models of the mechanics of the support of a system of rigid bodies with a tree structure, and efficient algorithms for the computer construction of them.

1°. No labour and time are needed to construct symbolic expressions for the kinetic and potential energies, or the Gauss and Appell etc. functions.

2°. No use is made in the coefficients of the models of differentiation operators (Jacobi matrices, Hessians, three-index Boltzmann symbols, Christoffel symbols of the first and second kind, etc.), so that they can be symbolically constructed without using differentiating modules of systems of analytic calculations.

3°. The algorithms are universal: they allow symbolic and numerical forms of the models of support mechanics to be constructed independently.

4°. The use of our algorithms reduces the construction of Eqs.(2.7) and (3.1) to the independent (sequential or parellel) formation of the elements of their matrix coefficients. This means that, in the symbolic constructions, we can perform these operations sequentially, using the same small block of the computer working memory, if the intermediate results are printed out (or are recorded in an external memory), or, in the numerical constructions, we can reduce the working time by making the computations in parallel form.

5°. The symbolic or numerical components of the vectors of the quasivelocities $V_p{}^{op}$, $V_p^{lk.p}$, used in the algorithms for fomring the matrices $\Phi_p{}^{op}$ and $\Phi_p^{lk.p}$, are obtained by means of a single economic recurrence algorithm (1.3) or relations (2.4).

$6^\circ$. The operations of multiplying a $(6\times 6)$ matrix by a $(6\times 1)$ column or $(1\times 6)$ row in the terms of Eqs.(2.7) and (3.1) breaks down into an operation of multiplying a $(3\times 3)$ matrix by a $(3\times 1)$ or $(1\times 3)$ vector if these matrices belong to one of four types, not counting the diagonal type (three elementary rotation matrices and a skew-symmetric matrix). This means that the strong matrix formalization of the models, which is convenient for using algebraic modules of systems of analytic calculations, can be combined with an economic form (in the computational sense) of the above operations. For this, we only need to introduce four computational algorithms which realize these operations as binary operations in $R_3$

$$
\begin{aligned}
c_1(\theta)\,x &= \|\, x_1,\ x_2\cos\theta - x_3\sin\theta,\ x_2\sin\theta + x_3\cos\theta \,\|^T\\
c_2(\theta)\,x &= \|\, x_1\cos\theta + x_3\sin\theta,\ x_2,\ x_3\cos\theta - x_1\sin\theta \,\|^T\\
c_3(\theta)\,x &= \|\, x_1\cos\theta - x_2\sin\theta,\ x_1\sin\theta + x_2\cos\theta,\ x_3 \,\|^T\\
\langle y\rangle\,x &= \|\, x_3y_2 - x_2y_3,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2 \,\|^T
\end{aligned}
$$

so that we avoid the need for access to two-dimensional blocks and for performing operations with zeros.

$7^\circ$. The algorithms are easy to use and are therefore suitable for a wide range of specialists.

**6. Example.** For the space apparatus (11), carrying an antenna (12) on a ball joint and a three-section manipulator (22), (23), (24), if we ignore rotating flywheels, we have, by (1.4),

$$
\begin{aligned}
&\mathbf{r}^{11} \equiv \mathbf{q}^{11} = \mathbf{q},\quad \mathbf{r}^{12} = \|\, p_1{}^{12},\ p_2{}^{12},\ p_3{}^{12},\ \theta_4{}^{12},\ \theta_5{}^{12},\\
&\theta_6{}^{12}\,\|^T,\ \mathbf{r}^{22} = \|\, p_1{}^{22},\ p_2{}^{22},\ p_3{}^{22},\ \theta_4{}^{22},\ 0,\ 0\,\|^T\\
&\mathbf{r}^{23} = \|\, 0,\ 0,\ 0,\ 0,\ \theta_5{}^{23},\ 0\,\|^T\\
&\mathbf{r}^{24} = \|\, 0,\ 0,\ 0,\ 0,\ 0,\ \theta_6{}^{24}\,\|^T
\end{aligned}
$$

The matrix coefficients in (3.1) are

$$
\begin{aligned}
&\mathbf{I}(\mathbf{q},\gamma) = \|\, I_4{}^{12} \mid I_5{}^{12} \mid I_6{}^{12} \mid I_4{}^{22} \mid I_5{}^{23} \mid I_6{}^{24} \,\|\\
&\mathbf{J}(\mathbf{q},\gamma,\mathbf{q}^{\cdot},\gamma^{\cdot}) = \|\, J_4{}^{12} \mid J_5{}^{12} \mid J_6{}^{12} \mid I_4{}^{22} \mid I_5{}^{23} \mid I_6{}^{24} \,\|\\
&I_\beta{}^{lk} = L_{lk}{}^{11} A_+{}^{lk} M_{lk}{}^{\mu.k-1} f_\beta{}^{lk},\quad M_{lk}{}^{\mu.k-1} f_\beta{}^{lk} \equiv f_\beta{}^{lk},\quad lk \neq 12\\
&J_\beta{}^{12} = L_{12}{}^{11}\,(K_+{}^{12} M_{12}{}^{11} + A_+{}^{12} M_{12}{}^{11\cdot})\, f_\beta{}^{12}\\
&J_\beta{}^{lk} = L_{lk}{}^{11} K_+{}^{lk} f_\beta{}^{lk};\quad \beta = 4, 5, 6;\quad lk = 22, 23, 24\\
&A_+{}^{24} = \Theta_{24}{}^{24},\quad A_+{}^{23} = \Theta_{23}{}^{23} + L_{24}{}^{23}\Theta_{24}{}^{24} L_{24}{}^{23,T}
\end{aligned}
$$

etc.; $K_+{}^{24} = K_{24}{}^{24.24}$, $K_+{}^{23} = K_{23}{}^{23.23} + L_{24}{}^{23} K_{24}{}^{24.24} L_{24}{}^{23,T}$ etc.;

$$
\begin{aligned}
&K_{lk}{}^{st.lk} = B_{lk}{}^{lk} + A_{lk}{}^{lk}\Phi_{lk}{}^{st.lk,T};\quad lk = 22, 23, 24,\qquad st = 11, 22\\
&K_{lk.}{}^{lk.lk} = B_{lk}{}^{lk};\quad lk = 11, 22, 23, 24.
\end{aligned}
$$

The matrices $\Phi_{lk}{}^{st.lk}$ $(st = 11, 22, 23;\ lk = 12, 22, 23, 24)$ are constructed from the components of the vectors (1.4):

$$
\begin{aligned}
&V_{12}^{11.12} = M_{12}{}^{11}\,\|\, f_4{}^{12} \mid f_5{}^{12} \mid f_6{}^{12} \,\|\,\|\, \theta_4^{12\cdot},\ \theta_5^{12\cdot},\ \theta_6^{12\cdot} \,\|^T\\
&V_{22}^{11.22} = M_{22}{}^{11} f_4{}^{22}\theta_4^{22\cdot},\ V_{23}^{11,23} = L_{23}^{22,T} V_{22}^{11.22} + f_5{}^{23}\theta_5^{23\cdot}, \ldots, V_{24}^{23.24} = f_6{}^{24}\theta_6^{24\cdot}
\end{aligned}
$$

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# THE MECHANISM OF THE HARD APPEARANCE OF A TWO-FREQUENCY OSCILLATION MODE IN THE CASE OF ANDRONOV-HOPF REVERSE BIFURCATION*

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The mapping of Poincaré secants is used to prove that a two-frequency oscillation mode (2-torus) can arise as a result of the hard loss of stability of the equilibrium state. A necessary condition for the transition is the presence close to the equilibrium state of a saddle periodic motion, the unstable manifold of which is attracted to the stationary manifold. At the instant when the cycle vanishes (Andronov-Hopf reverse bifurcation) a close-to-homoclinic situation arise, when the unstable separatrix of the stationary state returns to a small neighbourhood of it along a stable direction.

Sufficient conditions are found for the Poincaré mapping to have an invariant curve corresponding to the appearance of a 2-torus in the initial system of differential equations. The possible connection of this scenario of stationary state with torus transition with the observed /1, 2/ mixed convection in a vertical layer with wavy boundaries in the case of numerical simulation is discussed.

**1. Formulation of the problem.** We consider the system of differential equations

$$u^{\cdot} = F(u, \mu), \quad u \in R^n, \quad \mu \in [-\mu_0, \mu_0] \tag{1.1}$$

where $F$ is a $C^\infty$-smooth or analytic function of $u, \mu$. We assume that $F(0, 0) = 0$ and that, when the sign of $\mu$ changes, an Andronov-Hopf reverse bifurcation occurs in the system. Let the equilibrium state $O$ at $\mu = 0$ be a node with respect to the hyperbolic variable and an unstable non-hyperbolic focus in the central manifold.

In the simplest case $n = 3$, when there is just one hyperbolic variable $x$, a smooth replacement of the coordinates and time can be used in some domain of variables $\mu$ and $u$, where $|\mu|$ and $|u|$ are sufficiently small, to reduce system (1.1) to the form

$$\begin{aligned} &\rho^{\cdot} = \mu\rho + \rho^3 + a\rho^5, \quad \varphi^{\cdot} = \omega \\ &x^{\cdot} = -\lambda x + N(\rho, \varphi, x, \mu) \end{aligned} \tag{1.2}$$

where $\rho$ and $\varphi$ are polar coordinates in the central manifold; the function $N$ includes higher-order terms, and $N = 0$ for $x = 0$.

For $\mu > 0$ the system has an equilibrium state (CP) of saddle type. If $\mu < 0$ there is a stable CP and a saddle periodic motion $L_\mu$ branching from it at the point $\mu = 0$. Let $W_0^s(\mu)$

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